

Quark and Gluon Condensates in Isospin Matter

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Applying the Hellmann-Feynman theorem to a charged pion gas, the quark and gluon condensates at low isospin density are determined by precise pion properties. At intermediate density around $f_\pi^2 m_\pi$, from both the estimation for the dilute pion gas and the calculation with Nambu–Jona-Lasinio model, the quark condensate is strongly and monotonously suppressed, while the gluon condensate is enhanced and can be larger than its vacuum value. This unusual behavior of the gluon condensate is universal for Bose condensed matter of mesons. Our results can be tested by lattice calculations at finite isospin density.

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While the vacuum structure of Quantum Chromodynamics (QCD) is rather complicated, quark and gluon condensates may be sufficient to determine the spectral properties of many hadrons. The lowest dimensional condensates in vacuum are [1] $\langle \bar{q}q \rangle_0 \simeq -(250 \text{ MeV})^3$ and $\langle G^2 \rangle_0 \simeq (360 \text{ MeV})^4$ with the definition $G^2 \equiv \alpha_s / \pi G_{\mu\nu}^a G_a^{\mu\nu}$, where α_s is the QCD coupling, q the light quark field, and $G_{\mu\nu}^a$ the gluon field tensor. On the other hand, the in-medium behavior of the condensates are of great importance for us to understand how the hot and dense environment modifies the vacuum structure and the hadron properties. Especially, the quark condensate is the order parameter of chiral symmetry restoration, and the gluon condensate may be related to the deconfinement phase transition [2].

For systems at zero temperature but finite baryon density ρ_B , the ratio $R_q(\rho_B) \equiv \langle \bar{q}q \rangle_{\rho_B} / \langle \bar{q}q \rangle_0$ and the difference $D_g(\rho_B) \equiv \langle G^2 \rangle_{\rho_B} - \langle G^2 \rangle_0$ between the condensates in baryon matter and in vacuum can be expressed as [1]

$$\begin{aligned} R_q(\rho_B) &= 1 - \frac{\sigma_N}{f_\pi^2 m_\pi^2} \rho_B + \dots, \\ D_g(\rho_B) &= -\frac{8}{9} (m_N - \sigma_N - S) \rho_B + \dots, \end{aligned} \quad (1)$$

where m_π , f_π , σ_N and S are pion mass, pion decay constant, nucleon σ term and strangeness content of nucleon in vacuum, and \dots denotes the higher order correction. Taking only the linear terms of R_q and D_g which are valid at low density, the quark condensate at nuclear saturation density is 25 – 50% smaller than its vacuum value, but the gluon condensate is reduced by only 3 – 6%. The large uncertainty is from the σ term which is not yet precisely determined. To obtain the behavior of the condensates at high density, effective models may be used.

Recently, the study on QCD in medium is extended to finite isospin and strangeness densities [3, 4]. The physical motivation to discuss isospin matter is to understand the mechanism of QCD phase transitions at finite density. While there is not yet precise lattice result at finite baryon density due to the fermion sign problem, it is in principle no problem to do lattice simulation at finite isospin density [3]. The QCD phase structure in isospin matter is also investigated in many low energy effective models, such as chiral perturbation theory [4, 5], ladder QCD [6], random matrix method [7], strong

coupling lattice QCD [8], and Nambu–Jona-Lasinio (NJL) model [9, 10, 11, 12]. Very recently, the isospin matter is discussed in the frame of AdS/CFT [13]. In this paper, we study the quark and gluon condensates in a model independent way at low isospin density and compare the prediction with a model calculation at high isospin density. *While it is generally believed that the density effect will reduce the condensates, we will see that the gluon condensate at finite isospin density can be enhanced and even be larger than its vacuum value. Our result is qualitatively different from the case at finite baryon density, and the physical reason may be the repulsive meson-meson interactions.*

Since pions are the lightest excitations of QCD carrying isospin, the ground state of QCD at small isospin density can be considered as a dilute gas of charged pions. Without loss of generality, the matter is assumed to be composed of π^+ mesons with isospin density $\rho_I > 0$. The relevant $\pi - \pi$ interaction in such a system is characterized by the s-wave $\pi - \pi$ scattering length a in $I = 2$ channel. The value of a is predicted many years ago [14] and can be taken as $a = m_\pi / (16\pi f_\pi^2) \simeq 0.043 m_\pi^{-1}$.

At zero temperature and low isospin density, when the condition $\rho_I a^3 \ll 1$ is satisfied, the pion matter which is in Bose-Einstein condensation state can be described by the LHY (Lee, Huang and Yang) theory [15] which is accepted as a general theory for weakly interacting Bose gas [16]. The energy density of the system is written as

$$\mathcal{E} = m_\pi \rho_I + \frac{2\pi a \rho_I^2}{m_\pi} \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho_I a^3} + \dots \right). \quad (2)$$

The first term on the right hand side corresponds to the rest energy, and the second one is the interacting energy in LHY. Note that the leading order correction $2\pi a \rho_I^2 / m_\pi$ was predicted by Bogoliubov [17] before LHY. The pressure \mathcal{P} and isospin chemical potential μ_I can be easily obtained from \mathcal{E} ,

$$\begin{aligned} \mathcal{P} &= \frac{2\pi a \rho_I^2}{m_\pi} \left(1 + \frac{64}{15\sqrt{\pi}} \sqrt{\rho_I a^3} + \dots \right), \\ \mu_I &= m_\pi + \frac{4\pi a \rho_I}{m_\pi} \left(1 + \frac{32}{3\sqrt{\pi}} \sqrt{\rho_I a^3} + \dots \right). \end{aligned} \quad (3)$$

From these relations, one can determine the pion properties

with the equation of state of the system,

$$m_\pi = \lim_{\rho_1 \rightarrow 0} \mu_1(\rho_1), \quad a = \lim_{\rho_1 \rightarrow 0} \frac{m_\pi \mathcal{P}(\rho_1)}{2\pi\rho_1^2}. \quad (4)$$

The behavior of the in-medium quark and gluon condensates at low density can be derived in a model independent way. In the QCD Hamiltonian density \mathcal{H}_{QCD} , chiral symmetry is explicitly broken by the current quark mass term $\mathcal{H}_{\text{mass}} = 2m_q \bar{q}q + m_s \bar{s}s + \dots$, where s is the strange quark field, m_q and m_s are light and strange quark masses, and \dots denotes heavy quark (c,b,t) contribution which is irrelevant to our discussion. Possible isospin breaking effect is neglected here, since it will not change our result. According to the Hellmann-Feynman theorem of quantum mechanics [18], we obtain

$$2m_q \langle \psi | \int d^3x \bar{q}q | \psi \rangle = m_q \frac{d}{dm_q} \langle \psi | \int d^3x \mathcal{H}_{\text{QCD}} | \psi \rangle, \quad (5)$$

where $|\psi(m_q)\rangle$ is the ground state of the system as a function of light quark mass, and we have multiplied the equation by m_q to obtain renormalization-group invariant quantities.

Applying the Hellmann-Feynman theorem (5) to the isospin matter and to vacuum and taking into account the uniformity of the system, we have

$$2m_q (\langle \bar{q}q \rangle_{\rho_1} - \langle \bar{q}q \rangle_0) = m_q \frac{d\mathcal{E}}{dm_q}, \quad (6)$$

where the derivative $d\mathcal{E}/dm_q$ is taken at fixed density.

The gluon condensate can be obtained by considering the trace of the energy-momentum tensor $T_\mu^\mu = -9/8G^2 + 2m_q \bar{q}q + m_s \bar{s}s$, where we considered only the u , d and s quarks and neglected the heavy quark contribution. The difference between the expectation values of the trace of the energy-momentum tensor in isospin matter and in vacuum is $\langle T_\mu^\mu \rangle_{\rho_1} - \langle T_\mu^\mu \rangle_0 = \mathcal{E} - 3\mathcal{P}$, which leads to the following result for the change in the gluon condensate,

$$\langle G^2 \rangle_{\rho_1} - \langle G^2 \rangle_0 = -\frac{8}{9} [\mathcal{E} - 3\mathcal{P} - 2m_q (\langle \bar{q}q \rangle_{\rho_1} - \langle \bar{q}q \rangle_0)], \quad (7)$$

where we have ignored the strangeness content of pions, $m_s (\langle \bar{s}s \rangle_{\rho_1} - \langle \bar{s}s \rangle_0) = 0$.

Taking the Weinberg result $a \propto m_\pi/f_\pi^2$ [14] and neglecting the m_q -dependence of the decay constant f_π , we have $da/dm_\pi = a/m_\pi$. Since the energy density \mathcal{E} is only a function of m_π at fixed ρ_1 , the derivative in (6) can be expressed as $d\mathcal{E}/dm_q = (d\mathcal{E}/dm_\pi)(dm_\pi/dm_q)$. Combining with the Gellmann-Oakes-Renner relation $2m_q \langle \bar{q}q \rangle_0 = -m_\pi^2 f_\pi^2$ and the LHY energy density (2), we finally obtain the ratio $R_q(\rho_1) \equiv \langle \bar{q}q \rangle_{\rho_1} / \langle \bar{q}q \rangle_0$ and difference $D_g(\rho_1) \equiv \langle G^2 \rangle_{\rho_1} - \langle G^2 \rangle_0$ between the condensates in isospin matter and in vacuum,

$$R_q(\rho_1) = 1 - \frac{\rho_1}{2f_\pi^2 m_\pi} - \frac{64\sqrt{\pi}a^{5/2}}{5f_\pi^2 m_\pi^3} \rho_1^{5/2} + \dots, \quad (8)$$

$$D_g(\rho_1) = -\frac{8}{9} \left(\frac{m_\pi}{2} \rho_1 - \frac{4\pi a}{m_\pi} \rho_1^2 - \frac{64\sqrt{\pi}a^{5/2}}{3m_\pi} \rho_1^{5/2} + \dots \right).$$

The linear ρ_1 -dependence of R_q and D_g can be completely determined via only two parameters, the pion mass m_π and decay constant f_π . From the quark condensate, it is natural to define a density scale $\rho_0 = f_\pi^2 m_\pi$, which is approximately equal to the nuclear saturation density ρ_{sat} ($\rho_0 \simeq 1.1\rho_{\text{sat}}$), and the linear dependence is valid for $\rho_1 \ll \rho_0$. The higher order correction in isospin matter is quite different from the one in baryon matter. For the quark condensate, the leading order correction in isospin matter is $O(\rho_1^{5/2})$, while it is $O(\rho_B^{4/3})$ [1] in baryon matter due to its fermionic nature. For the gluon condensate, since the coefficient $m_\pi/2$ of the linear term in isospin matter is much less than the one ($m_N - \sigma_N - S$) in baryon matter and the $\pi - \pi$ interaction is repulsive, the competition between the linear and leading terms may make the condensate to increase at high isospin density. Neglecting the term $O(\rho_1^{5/2})$ which is indeed small even at high density and using the Weinberg result $a = m_\pi/(16\pi f_\pi^2)$ [14], the gluon condensate starts to increase at $\rho_1 = \rho_0$ and becomes larger than its vacuum value at $\rho_1 > 2\rho_0$. While this prediction is beyond the validity density region $\rho_1 \ll \rho_0$ of the LHY equation of state, we do expect that the gluon condensate may go beyond its vacuum value at high isospin density. We will examine it in the following with a chiral quark model.

To estimate the behavior of the quark and gluon condensates at high density is beyond the above approach, since the composite nature of pions may become important at high isospin density. Especially, at extremely high ρ_1 , the isospin matter is expected to become a weakly coupled Fermi superfluid [3]. In this case, the element constitutes of the system are no longer pions but quarks. To have a complete understanding of the quark and gluon condensates at finite isospin density, we adopt an effective chiral model at quark level to describe the evolution from a weakly interacting Bose condensate to a Fermi superfluid.

One of the models that enables us to see directly how the dynamic mechanism of chiral symmetry breaking and restoration operate is the NJL model [19] applied to quarks [20]. Recently, this model is extended to finite isospin chemical potential [9, 10, 11, 12]. The Lagrangian density of the model is

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m_q) \psi + G \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2 \right], \quad (9)$$

where $\psi = (u, d)$ is the two-flavor quark field. The current quark mass m_q , the coupling constant G and a high momentum cutoff Λ due to the non-renormalization of the model are phenomenological parameters and can be determined by fitting the pion mass, pion decay constant and quark condensate in vacuum. The isospin density enters the model via introducing an isospin chemical potential μ_1 , corresponding to the isospin charge $I = \int d^3x \bar{\psi}\gamma_0\tau_3\psi$. The order parameters for chiral symmetry breaking and isospin symmetry breaking are respectively the quark condensate $\langle \bar{q}q \rangle$ and pion condensate $\langle \bar{u}i\gamma_5 d \rangle$.

At mean field level, the thermodynamic potential of the

isospin matter in the NJL model can be evaluated as [12]

$$\Omega = -6 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(E_+ + E_- - 2\sqrt{\mathbf{k}^2 + M_0^2} \right) + \frac{(M - m_q)^2 - (M_0 - m_q)^2 + \Delta^2}{4G} \quad (10)$$

with the dispersions $E_{\pm} = \sqrt{(\xi_{\pm})^2 + \Delta^2}$, $\xi_{\pm} = \epsilon \pm \mu_1/2$ and $\epsilon = \sqrt{\mathbf{k}^2 + M^2}$, where $M = m_q - 4G\langle\bar{q}q\rangle$ is the in-medium quark mass, M_0 is its vacuum value, and $\Delta = -4G\langle\bar{u}i\gamma_5 d\rangle$ is a BCS-like energy gap. The physical quark and pion condensates correspond to the minimum of the thermodynamic potential,

$$\frac{\partial\Omega}{\partial M} = 0, \quad \frac{\partial\Omega}{\partial\Delta} = 0, \quad (11)$$

which together with the isospin density

$$\rho_1 = -\frac{\partial\Omega}{\partial\mu_1} \quad (12)$$

determine self-consistently M , Δ and μ_1 as functions of ρ_1 .

From the above gap and density equations, a nonzero isospin density is associated with a nonzero pion condensate [12]. In the limit of $\rho_1 \rightarrow 0$, we have analytically $\mu_1 \rightarrow m_\pi$. In Fig.1, we show the numerically calculated isospin chemical potential μ_1 and pressure $\mathcal{P} = -\Omega$ in the NJL model and compare them to the LHY result (3) with the standard Weinberg value $a = m_\pi/(16\pi f_\pi^2)$. Note that the higher order correction $O(\sqrt{\rho_1 a^3})$ in LHY is too small to be observed when the isospin density is not high enough. We find a very good agreement of the two calculations at sufficiently low density. At $\rho_1 \sim 0.1\rho_0$, the composite nature of pions emerges and the NJL model starts to deviate from the LHY result. Note that for baryon matter, the NJL model agrees with the low-density result (1) only for some special model parameters.

Having the correct low density limit, we then consider the quark and gluon condensates at finite isospin density. The quark condensate is obtained via either directly solving the NJL gap equations (11) or the Hellmann-Feynman result (6) through the NJL energy density $\mathcal{E} = -\mathcal{P} + \mu_1\rho_1$, and the gluon condensate can be calculated via Eq. (7) together with the Hellmann-Feynman result (6). The numerical results from the LHY theory and NJL model are shown in Fig.2 for ρ_1 up to $2\rho_0$. At very low density, the NJL model agrees well with the model independent result, but at intermediate density, the LHY theory for the dilute pion gas fails and the difference between the two results becomes significant. The quark condensate decreases much faster than the gluon condensate. At $\rho_1 = \rho_0$, the change in the gluon condensate is very small, but the quark condensate has already been reduced by about 45%, which is almost the same as the condensate suppression in baryon matter at nuclear saturation density. The other significant characteristic of the gluon condensate is its non-monotonous density dependence. While this property is outside the validity density region of the LHY theory, it is confirmed in the NJL model. The gluon condensate firstly drops

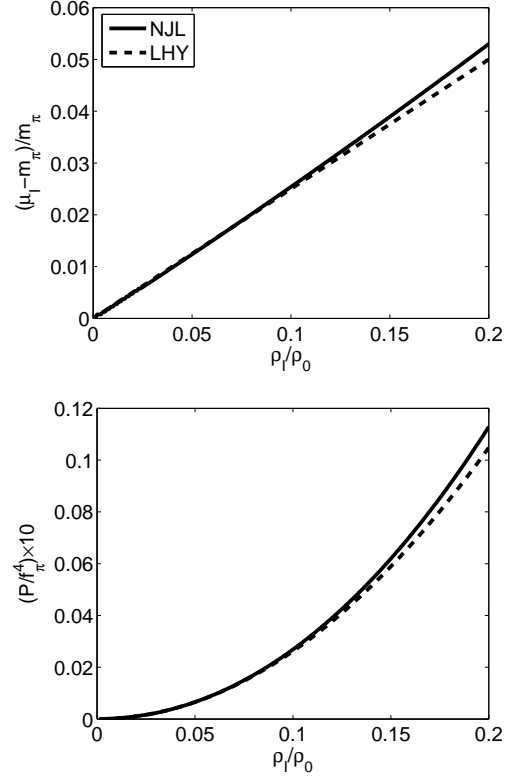


FIG. 1: The isospin chemical potential μ_1 and pressure \mathcal{P} at low isospin density, calculated with the NJL model and LHY theory.

down, then turns to go up at $\rho_1 \simeq 0.6\rho_0$ in the NJL model and $\rho_1 \simeq \rho_0$ in the LHY theory, and finally becomes larger than its vacuum value at $\rho_1 > 1.14\rho_0$ and $\rho_1 > 2\rho_0$ in the two calculations.

The strong suppression of the quark condensate and the enhancement of gluon condensate at intermediate density will induce the evolution from a weakly interacting Bose condensate to a strongly interacting Fermi superfluid, i.e., the so-called BEC-BCS crossover [21]. From the fermion excitation spectrum, a Fermi surface is opened with $\mu_1/2 > M$ [22] at $\rho_1 > 1.6\rho_0$. This type of strongly interacting Fermi superfluid may be a realization of the quarkyonic matter proposed recently [23]. In such a matter, the chiral symmetry is approximately restored, but the quarks are still confined. Our prediction on the non-monotonous gluon condensate in isospin matter is consistent with the expectation that the hadron and quark phases are continued and there may exist no deconfinement phase transition at zero temperature [3].

In summary, we have investigated the quark and gluon condensates in isospin matter. At low isospin density, we derived the model-independent condensates by taking the Hellmann-Feynman theorem for a dilute pion gas. Unlike the baryon matter, the condensates in isospin matter are determined by precise pion properties. Both the low density formula and the NJL model calculation show that the repulsive interactions between the charged pions may induce an unusual behavior of

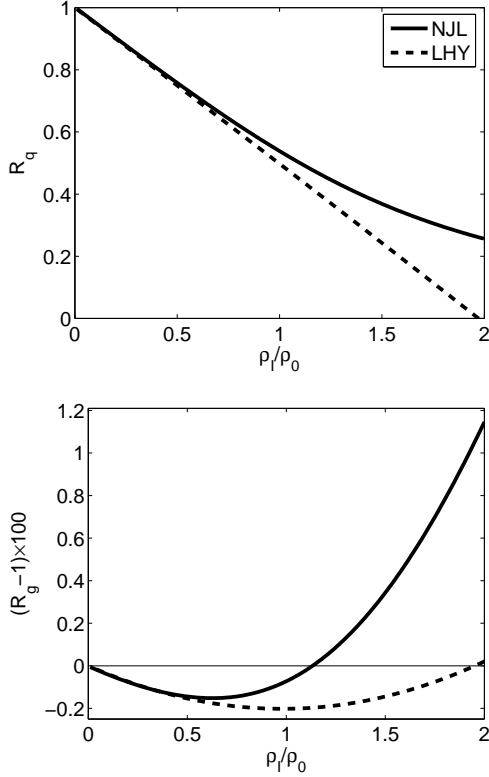


FIG. 2: The condensate ratios $R_q = \langle \bar{q}q \rangle_{\rho_I} / \langle \bar{q}q \rangle_0$ and $R_g = \langle G^2 \rangle_{\rho_I} / \langle G^2 \rangle_0$ from low to intermediate isospin density, calculated in the NJL model and LHY theory with $\langle G^2 \rangle_0 = (360 \text{ MeV})^4$.

the gluon condensate: it decreases slightly at low density and turns to increase at intermediate density. While there are no lattice data of the gluon condensate at finite isospin density, we expect our conclusion can be examined in the future lattice calculations.

Our calculation for isospin matter can be easily extended to QCD at finite isospin density ρ_I and strangeness density ρ_S which is associated with kaon condensation [4]. For kaon matter with $\rho_S = 2\rho_I$, taking into account the relation $m_K^2 f_K^2 = -(m_s + m_q)(\langle \bar{s}s \rangle_0 + \langle \bar{q}q \rangle_0)/2$ for the strangeness condensate $\langle \bar{s}s \rangle_0$ in vacuum and the kaon mass m_K and kaon decay constant f_K , while the light and strange quark condensates behave differently, the gluon condensate in (8) for pion matter is valid for kaon matter, if we replace m_π , ρ_I and a by m_K , ρ_S and kaon-kaon scattering length a_K in $I = 1$ channel, $D_g(\rho_S) = -8/9 (m_K \rho_S/2 - 4\pi a_K \rho_S^2/m_K + \dots)$. Taking the recent lattice QCD result $m_K a_K = 0.352$ [24], the turning density where the gluon condensate starts to increase is $\rho_S \simeq f_K^2 m_K$ for kaon matter which is the same as $\rho_I \simeq f_\pi^2 m_\pi$ for pion matter.

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